**ADDITIONAL COLLABORATION IDEAS INVOLVING GAP**

## 1. Seven ideas

(1) Let $\mathcal{O}_d$ denote the ring of integers of $\mathbb{Q}(\sqrt{d})$. Bergeron and Venkatesh [1] have conjectured that for $d < 0$ we have

$$\frac{\log |H_1(\Gamma_0(N), \mathbb{Z})_{tors}|}{\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H}^3)} \to \frac{1}{6\pi}$$

as $N$ tends to infinity among primes, where $\Gamma_0(N) < SL_2(\mathcal{O}_d)$ and

$$\mathrm{vol}(SL_2(\mathcal{O}_d)\backslash\mathbb{H}^3) = \frac{|D|^{3/2}}{24}\zeta_{\mathbb{Q}(\sqrt{d})}(2)/\zeta_{\mathbb{Q}}(2)$$

with $D$ the discriminant of $\mathbb{Q}(\sqrt{d})$. Investigate this computationally (*cf.* Example 11).

(2) Schwermer and Vogtmann [5] have calculated $H_*(SL_2(\mathcal{O}_d), \mathbb{Z})$ for the euclidean cases $d = -1, -2, -3, -7, -11$. Compute this homology for the remaining imaginary quadratic principal ideal domains. Schwermer and Vogtmann also calculate $H_*(PSL_2(\mathcal{O}_d), \mathbb{Z})$ for the euclidean cases. Recompute their $PSL_2$ result for the case $d = -7$. Compute the $PSL_2$ homology in low degrees for the remaining imaginary quadratic principal ideal domains and guess a formula for arbitrary degree (*cf.* Example 12).

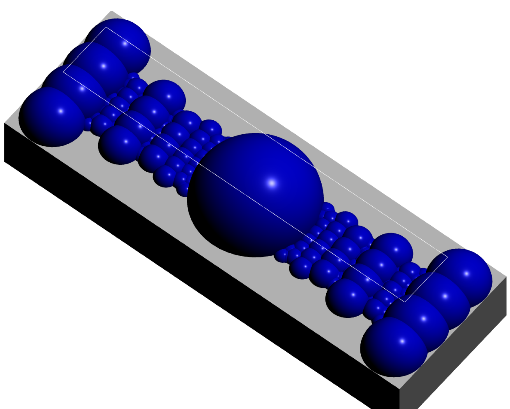
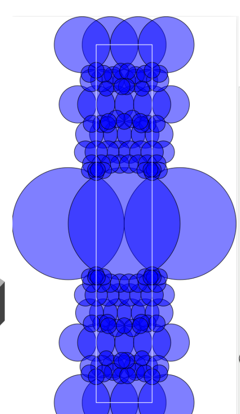
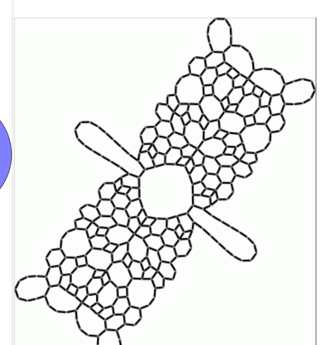
(3) Swan [6] produced a finite presentation for $SL_2(\mathcal{O}_d)$ in the cases $d = -1, -2, -3, -7, -11, -19$ (as well as in some multiply cusped cases) and used the presentation to compute the abelianization $SL_2(\mathcal{O}_d)_{ab}$. He wrote:

> **The calculations obviously have not been pushed far enough to make any reasonable conjecture about the rank of $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})]$. The length of the calculation increases so rapidly with the discriminant that machine computation seems to be the only reasonable approach.**

Presentations and abelianizations for the remaining principal ideal cases have been given by Reese [4]. Yasaki [7] describes a different method for finding presentations, based on the Voronoi complex, and illustrates it for $d = -14$. Pursue Swan's comment quoted above (*cf.* Example 14).

(4) Investigate $GL_2(\mathcal{O}_d)_{ab}$ in the real quadratic case $d > 0$ (see Example 15 which requires Magma to be installed).

(5) The following representations of the 'bottom' 2-cells in the boundary of a fundamental domain for $SL_2(\mathcal{O}_{-163})$ were produced using Example 13.

The first two images are produced using the Asymptote vector graphics language and the third uses Graphviz to represent the 1-skeleton as a combinatorial graph. Improve these images. [For instance, combine the second and third images into a single image giving a geometric representation of the projection of the 2-cells onto the complex plane. Maybe the first image would be embellished by printing a (suitably symmetric) fish on each 2-cell, the colour of the fish representing the orbit of the cell and the directions of fish in a given orbit somehow representing the action of $SL_2(\mathcal{O}_{-163})$?]

(6) Browder and Pakianathan [2] considered the group $G_p = \ker(SL_2(\mathbb{Z}_p^3) \to SL_2(\mathbb{Z}_p))$ of order $p^6$ and exponent $p^2$. They proved that $H_*(G_p, \mathbb{Z})$ has a certain exponent $e$ where $p > 2$ is a prime. For the case $p = 3$ use GAP to guess $e$ and to prove a lower bound $e_L \leq e$ and upper bound $e_U \geq e$. I could only achieve $e_U - e_L = 1$ but maybe you can do better. (*cf.* Example 4.)

(7) Milgram [3] wrote:

> and, as a result of our calculation $H_i(M_{23}; \mathbb{Z}) = 0$ for $i < 5$. In particular $M_{23}$ is the first known counterexample to the conjecture that if $G$ is a finite group with $H_i(G; \mathbb{Z}) = 0$, $i = 1, 2, 3$, then $G = \{1\}^{\ddagger}$. ($M_{11}$, the first Mathieu group and $J_1$, the first Janko group also satisfy $Out(G) = Mult(G) = 1$, but for both of these groups $H_3(G; \mathbb{Z}) \neq 0$.) It would be tempting to amend the conjecture. It is very likely that it only fails for a very small number of the sporadics among the simple groups. So one might well suspect that there is a (small) finite number $n$ so that $H^i(G; \mathbb{Z}) = 0$ for $0 < i \leq n$ implies that $G = \{1\}$ if $G$ is finite. But I have no idea as to a suitable candidate for $n$.
>
> ‡ This conjecture is orginally due to Loday in the mid 1970's and is based on Quillen's calculations of the cohomology of many of the families of groups of Lie type a few years earlier, (C. Giffen, [G]).

Use GAP to compute the first few integral homology groups of $M_{23}$. My laptop, with the default algorithm for generic finite groups, gets as far as homology degree $i = 7$. (Techniques involving the Quillen complex can be used to obtain further information on the integral homology.) Then use GAP's library of perfect groups to construct a list of perfect groups with trivial second integral homology. Do any of the groups in your list have trivial third integral homology? (*cf.* Example 4.)

## 2. Fifteen examples

The examples below and the online HAP manual should be of help. For the examples to run you'll need to create a directory /pkg and download the current state of the HAP repository to it by typing:

```
mkdir pkg
cd pkg
git clone https://github.com/gap-packages/hap.git
```

To start GAP with the current state of HAP then type:

```
gap -l '~;'
```

The following examples should then work. Commands involving visualization require that graphviz and asymptote are installed. Some commands involving crystallographic groups require that polymake is installed. If an example seems too slow then it would be great if you could try to identify the bottlenecks and formulate suggestions for improving them.

```
#Example 1
G:=SpaceGroup(3,9);;
v:=[1/2,1/3,1/5];;
K:=CrystallographicComplex(G,v);;
Polymake(K!.fundamentalDomain,"VISUAL");;
K!.dimension(0);
K!.dimension(1);
K!.dimension(2);
K!.dimension(3);
K!.stabilizer(0,1);
Order(K!.stabilizer(0,1));
Order(K!.stabilizer(1,1));
R:=FreeGResolution(K,4);
Cohomology(HomToIntegers(R),3);
Homology(TensorWithIntegers(R),3);
v:=[0,0,0];;
K:=CrystallographicComplex(G,v);;
Polymake(K!.fundamentalDomain,"VISUAL");;


#Example 2
G:=SpaceGroup(3,8);;
K:=CrystallographicComplex(G);;
Polymake(K!.fundamentalDomain,"VISUAL");
K!.dimension(0);
K!.dimension(1);
K!.dimension(2);
K!.dimension(3);
Order(K!.stabilizer(0,1));
Order(K!.stabilizer(1,1));
R:=FreeGResolution(K,10);
K!.dimension(0);R!.dimension(0);
K!.dimension(1);R!.dimension(1);
K!.dimension(2);R!.dimension(2);
K!.dimension(3);R!.dimension(3);
K!.dimension(4);R!.dimension(4);
Cohomology(HomToIntegers(R),3);
Homology(TensorWithIntegers(R),9);


#Example 3
G:=SpaceGroup(3,8);;
R:=ResolutionCubicalCrystGroup(G,100);
IntegralRingGenerators(R,0);
IntegralRingGenerators(R,1);
IntegralRingGenerators(R,2);
IntegralRingGenerators(R,3);
IntegralRingGenerators(R,4);
IntegralRingGenerators(R,5);
IntegralRingGenerators(R,6);
IntegralRingGenerators(R,7);
IntegralRingGenerators(R,8);
```

```
#Example 4
GroupHomology(MathieuGroup(23),1);
GroupHomology(MathieuGroup(23),2);
GroupHomology(MathieuGroup(23),3);
Order(MathieuGroup(23));
GroupHomology(MathieuGroup(23),4);


#Example 5
H:=SylowSubgroup(MathieuGroup(24),2);
C:=CompositionSeries(H);;
R:=ResolutionSubnormalSeries(C,4);;
P:=PresentationOfResolution(R);;
P.freeGroup/P.relators;
P.relators;
g:=Random(H);
P.wordInFreeGenerators(g);


Homology(TensorWithIntegers(R),3);
gens:=GeneratorsOfGroup(AutomorphismGroup(H));;
eqmap:=EquivariantChainMap(R,R,gens[1]);
f:=Homology(TensorWithIntegers(eqmap),2);
Order(f);Order(gens[1]);


#Example 6
P:=BianchiPolyhedron(-5);
Display3D(P);
K:=BianchiGcomplex(-5);
K!.dimension(0);
K!.dimension(1);
K!.dimension(2);
R:=FreeGResolution(K,20);
Homology(TensorWithIntegers(R),1);
Homology(TensorWithIntegers(R),2);
Homology(TensorWithIntegers(R),3);
Homology(TensorWithIntegers(R),4);
Homology(TensorWithIntegers(R),5);
Homology(TensorWithIntegers(R),6);
Homology(TensorWithIntegers(R),7);
Homology(TensorWithIntegers(R),8);
Homology(TensorWithIntegers(R),9);
Homology(TensorWithIntegers(R),10);



#Example 7
K:=BianchiGcomplex(-5);;

chi:=0;;
for n in [0..2] do
for k in [1..K!.dimension(n)] do
g:=Order(K!.stabilizer(n,k));
if g < infinity then chi:=chi + (-1)^n/g; fi;
```

```
od;od;
chi;


#Example 8
V:=VoronoiComplexSL(2,-5);;
R:=FreeGResolution(V,5);;
Homology(TensorWithIntegers(R),2);

M:=HomogeneousPolynomials(R!.group,8);;
C:=HomToIntegralModule(R,M);;
Cohomology(C,4);


#Example 9
gamma:=HAP_CongruenceSubgroupGamma0(11);;
k:=4;; deg:=1;; c:=CuspidalCohomologyHomomorphism(gamma,deg,k);;
AbelianInvariants(Kernel(c));


#Example 10
R:=ResolutionSL2Z_alt(2);;
G:=R!.group;;
P:=HomogeneousPolynomials(G,10);;
Cohomology(HomToIntegralModule(R,P),1);
#The space S_12 of cusp forms is of dimension 1


G:=HAP_CongruenceSubgroupGamma0(1);;
eigs:=[];;
for p in [2,3,5,7,11,13,17,19] do
T:=HeckeOperator(G,p,12);;
Print("eigenvalues= ",Eigenvalues(Rationals,T),
" and eigenvectors = ", Eigenvectors(Rationals,T)," for p= ",p,"\n\n");
Add(eigs,Eigenvalues(Rationals,T)[2]);
od;


q:=Indeterminate(Integers,"q");
f:=q*Product(List([1..20], n->(1-q^n)^24));;
c:=CoefficientsOfUnivariatePolynomial(f);;

eigs;
c{[2,3,5,7,11,13,17,19]+1};


gap> #Example 11
gap> Q:=QuadraticNumberField(-1);;
gap> OQ:=RingOfIntegers(Q);;
gap> I:=QuadraticIdeal(OQ,41+56*Sqrt(-1));
gap> IsPrime(I);
gap> G:=HAP_CongruenceSubgroupGamma0(I);;
gap> R:=ResolutionSL2QuadraticIntegers(-1,2,true);;
gap> S:=ResolutionFiniteSubgroup(R,G);; #free resolution for G of length 2
gap> C:=TensorWithIntegers(S);;
```

```
gap> C:=ContractedComplex(C);; #This should help the SNF algorithm
gap> h:=Homology(C,1);
gap> Loge10:=0.434294481903;; #Log_10(e)
gap> 1.0*Log(Product(h),10)/(Loge10*Norm(I));


#Example 12
gap> R:=ResolutionSL2QuadraticIntegers(-1,10);;
gap> Homology(TensorWithIntegers(R),8);
gap> R:=ResolutionPSL2QuadraticIntegers(-1,10);;
gap> Homology(TensorWithIntegers(R),8);


gap> #Example 13
gap> P:=BianchiPolyhedron(-163);;
gap> Display3D(P);;
gap> Display2D(P);;
gap> Y:=RegularCWComplex(P);;
gap> Display(GraphOfRegularCWComplex(Y));


gap> #Example 14
gap> K:=BianchiGcomplex(-5);; #Swan's method
gap> R:=FreeGResolution(K,2);;
gap> P:=PresentationOfResolution(R);;
gap> G:=P.freeGroup/P.relators;; #presentation for SL(2,O-5)
gap> AbelianInvariants(G);
gap> RelatorsOfFpGroup(G);
gap> G:=SimplifiedFpGroup(G);;
gap> RelatorsOfFpGroup(G);
gap> K!.generators; #generators used by Swan


gap> V:=VoronoiComplexSL(2,-5);; #Voronoi method (needs Magma; see HAP manual)
gap> R:=FreeGResolution(V,2);;
gap> P:=PresentationOfResolution(R);;
gap> G:=P.freeGroup/P.relators;; #presentation for SL(2,O-5)
gap> AbelianInvariants(G);


gap> #Example 15
gap> V:=VoronoiComplexGL(2,5);; #Voronoi method (needs Magma)
gap> R:=FreeGResolution(V,2);;
gap> P:=PresentationOfResolution(R);;
gap> G:=P.freeGroup/P.relators;; #presentation for GL(2,O5)
gap> AbelianInvariants(G);
```

## References


[1] Nicolas Bergeron and Akshay Venkatesh. The asymptotic growth of torsion homology for arithmetic groups. *J. Inst. Math. Jussieu*, 12(2):391–447, 2013.

[2] William Browder and Jonathan Pakianathan. Cohomology of uniformly powerful $p$-groups. *Trans. Amer. Math. Soc.*, 352(6):2659–2688, 2000.

[3] R. James Milgram. The cohomology of the Mathieu group $M_{23}$. *J. Group Theory*, 3(1):7–26, 2000.

[4] Tanner Reese. Presentations for singly-cusped Bianchi groups. *Topology and its Applications*, 328:108443, 2023.

[5] Joachim Schwermer and Karen Vogtmann. The integral homology of $SL_2$ and $PSL_2$ of euclidean imaginary quadratic integers. *Commentarii Mathematici Helvetici*, 58(1):573–598, Dec 1983.

[6] Richard G. Swan. Generators and relations for certain special linear groups. *Advances in Math.*, 6:1–77 (1971), 1971.

[7] Dan Yasaki. Hyperbolic tessellations associated to Bianchi groups. In *Algorithmic number theory*, volume 6197 of *Lecture Notes in Comput. Sci.*, pages 385–396. Springer, Berlin, 2010.